COMPLEX K3 SURFACES CONTAINING LEVI-FLAT HYPERSURFACES

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Abstract

We show the existence of a complex K3 surface X which is not a Kummer surface and has a one-parameter family of Levi-flat hypersurfaces in which all the leaves are dense. We construct such X by patching two open complex surfaces obtained as the complements of tubular neighborhoods of elliptic curves embedded in blow-ups of the projective planes at general nine points.

1 Introduction

A real hypersurface M in a complex manifold X is said to be Levi-flat if it admits a foliation of real codimension 1 whose leaves are complex manifolds holomorphically immersed into X. We shall show the following:

Theorem 1.1. *There exists a K3 surface X which is not a Kummer surface and has a one-parameter family of Levi-flat hypersurfaces in which all the leaves are dense.*

More precisely, we will construct a K3 surface (i.e. a compact simply-connected 2-dimensional complex manifold with trivial canonical bundle) X which is not a Kummer surface and has an open complex submanifold $V \subset X$ with the following property: there exists an elliptic curve E, a non-torsion flat line bundle $F \to E$, and two positive numbers $a < b$ such that V is biholomorphic to $\{x \in F \mid a < |x|_h < b\}$, where h is a fiber metric on F with zero curvature. The Levi-flat hypersurfaces in Theorem 1.1 are given by the hypersurfaces corresponding to $\{x \in F \mid |x|_h = t\}$ for $a < t < b$. Note that Kummer surfaces with such an open subset V can be constructed in a simple manner. Actually, the

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Kummer surface constructed from an abelian surface A has such V as an open complex submanifold if A includes V.

Each leaf of the Levi-flat hypersurface $\{x \in F \mid |x|_h = t\}$ is biholomorphic to the complex plane $\mathbb{C}$ or $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ for each $a < t < b$. Therefore, by considering the universal covering of a leaf, we have the following:

Corollary 1.2. *There exists a K3 surface X which is not a Kummer surface and admits a holomorphic map $f\colon \mathbb{C} \to X$ such that the Euclidean closure of the image $f(\mathbb{C})$ is a compact real hypersurface of X. In particular, the Zariski closure of $f(\mathbb{C})$ coincides with X, whereas the Euclidean closure of $f(\mathbb{C})$ is a proper subset of X.*

In the present paper, we will construct a K3 surface X by patching two open complex surfaces obtained as the complements of tubular neighborhoods of elliptic curves embedded in blow-ups of the projective planes $\mathbb{P}^2$ at general nine points. The outline of the construction is as follows: Let S be the blow-up of $\mathbb{P}^2$ at certain nine points $Z := \{p_1, p_2, \dots, p_9\} \subset \mathbb{P}^2$ and $C \subset S$ be the strict transform of the elliptic curve in $\mathbb{P}^2$ which includes Z. Denote by M the complement of a tubular neighborhood of C in S. Let S', C', and M' be those constructed from another appropriate nine points configuration $Z' := \{p'_1, p'_2, \dots, p'_9\} \subset \mathbb{P}^2$. We construct X by patching M and M'. In order to patch them holomorphically, one needs to choose Z and Z' in a suitable manner. For this purpose, we show the following:

Theorem 1.3. *Let C, S, M, C', S', and M' be as above. Assume that C and C' are biholomorphic, the normal bundles $N_{C/S}$ of C and $N_{C'/S'}$ of C' satisfy $N_{C/S} \cong N_{C'/S'}^{-1}$, and that $N_{C/S}$ satisfies the Diophantine condition. Then one can patch M and M' holomorphically. The resulting complex surface is a K3 surface.*

See §2.3 for the Diophantine condition. We apply Arnol'd's Theorem [A] on a neighborhood of an elliptic curve to show Theorem 1.3. This patching construction based on Arnol'd's Theorem is also used in [T] to study complex structures on $S^3 \times S^3$. Note that our construction can be regarded as a special case of the *gluing construction* studied in [D] (see [D, Example 5.1]). A main difference between our construction and the gluing construction by Doi is that, in our construction, one need not to deform the complex structures of M and M' to patch together. In particular, one can regard M and M' as holomorphically embedded open complex submanifolds of the resulting K3 surface in our construction.

The organization of the paper is as follows. In §2, we collect some fundamental facts on the Picard variety of an elliptic curve, the cohomology $H^q(S, T_S)$ of a blow-up S of $\mathbb{P}^2$, and Arnol'd's theorem. In §3, we show Theorem 1.3 and explain the details of the construction of X. In §4, we investigate the deformation of X. In §5, we show that X

is not a Kummer surface if one choose Z and Z' appropriately. Here we show Theorem 1.1 and Corollary 1.2. In §6, we show a relative variant of Arnol'd's Theorem, which is needed in §4.

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2 preliminaries

2.1 The universal line bundle on $C \times \mathrm{Pic}^0(C)$

Let C be a smooth elliptic curve. Fix a base point $p \in C$. In this subsection, we identify $\mathrm{Pic}^0(C)$ with C via the isomorphism $C \ni q \mapsto \mathcal{O}_C(q-p) \in \mathrm{Pic}^0(C)$. Denote by D_1 the prime divisor $\{(q,q) \in C \times C \mid q \in C\}$ and by D_2 the prime divisor $\{p\} \times C$ of $C \times \mathrm{Pic}^0(C) = C \times C$. Set $\mathcal{L} := \mathcal{O}_{C\times C}(D_1 - D_2)$ and regard it as a line bundle on $C \times \mathrm{Pic}^0(C)$.

Proposition 2.1. *Let T be a complex manifold and $\mathcal{N}$ be a holomorphic line bundle on $C \times T$. Assume that $\mathcal{N}|_{C\times\{t\}}$ is flat (i.e. $\mathcal{N}|_{C\times\{t\}} \in \mathrm{Pic}^0(C\times\{t\})$) for all $t \in T$. Then, there uniquely exists a holomorphic map $i\colon T \to \mathrm{Pic}^0(C)$ such that $(\mathrm{id}_C \times i)^*\mathcal{L} = \mathcal{N}$.*

Proof. As the map i needs to map a point $t \in T$ to the point which corresponds to $\mathcal{N}|_{C\times\{t\}}$, the uniqueness is clear. Therefore, all we have to do is to show the existence of such a holomorphic map i. It is sufficient to construct this map i by assuming T is a sufficiently small open ball centered at $0 \in \mathbb{C}^n$. In what follows, we denote by C_t the submanifold $C \times \{t\}$ and by N_t the line bundle $N_t := \mathcal{N}|_{C_t}$ for each $t \in T$. Fix $q_0 \in C$ such that $N_0 = \mathcal{O}_{C_0}(q_0 - p_0)$, where $p_0 := (p, 0)$. Consider the restriction map $H^0(C \times T, \mathcal{N} \otimes \mathrm{Pr}_1^*\mathcal{O}_C(p)) \to H^0(C_0, N_0 \otimes \mathcal{O}_{C_0}(p_0)) = H^0(C_0, \mathcal{O}_{C_0}(q_0))$, where $\mathrm{Pr}_1\colon C \times T \to C$ is the first projection. As it is easily observed, this map is surjective (Use, for example, Nadel's vanishing theorem to $H^1(C \times T, \mathcal{O}_{C\times T}(-C_0) \otimes \mathcal{N} \otimes \mathrm{Pr}_1^*\mathcal{O}_C(p))$). Therefore, there exists a holomorpchic section $F\colon C \times T \to \mathcal{N} \otimes \mathrm{Pr}_1^*\mathcal{O}_C(p)$ such that the zero divisor of $F|_{C_0}\colon C_0 \to \mathcal{O}_{C_0}(q_0)$ is equal to $\{q_0\}$. This means that the zero divisor $D := \mathrm{div}(F)$ of F transversally intersects C_0 at only the point q_0. Thus we may assume that D is a prime divisor and transversally intersects C_t at only one point, say $q_t \in C_t$, by shrinking T if

necessary. By the implicit function theorem, the map $t \mapsto q_t$ defines a holomorphic map $i\colon T \to C$. As it hold as divisors that $(\mathrm{id}_C \times i)^* D_1 = D$ and $(\mathrm{id}_C \times i)^* D_2 = \{p\} \times T$, the proposition follows. $\square$

In what follows, we call this line bundle $\mathcal{L}$ *the universal line bundle* on $C \times \mathrm{Pic}^0(C)$.

2.2 The cohomology of the tangent bundle of a blow-up of $\mathbb{P}^2$ at general points

Fix an integer $N \geq 4$ and distinct N points $Z := \{p_1, p_2, \dots, p_N\}$ in $\mathbb{P}^2$. Denote by S the blow-up of $\mathbb{P}^2$ at Z. In this subsection, we compute the cohomology groups $H^q(S, T_S)$, where T_S is the tangent bundle of S. By the simple computation, we obtain the short exact sequence $0 \to \pi_* T_S \to T_{\mathbb{P}^2} \to j_* N_{Z/\mathbb{P}^2} \to 0$, where $\pi\colon S \to \mathbb{P}^2$ is the blow-up morphism and $j\colon Z \to \mathbb{P}^2$ is the inclusion. This short exact sequence induces the following long exact sequence

$$\begin{aligned} (1) \qquad 0 \to\ & H^0(\mathbb{P}^2, \pi_* T_S) \to H^0(\mathbb{P}^2, T_{\mathbb{P}^2}) \to H^0(\mathbb{P}^2, j_* N_{Z/\mathbb{P}^2}) \\ \to\ & H^1(\mathbb{P}^2, \pi_* T_S) \to H^1(\mathbb{P}^2, T_{\mathbb{P}^2}) \to 0 \to H^2(\mathbb{P}^2, \pi_* T_S) \to H^2(\mathbb{P}^2, T_{\mathbb{P}^2}). \end{aligned}$$

From this exact sequence, we have the following:

LEMMA 2.2. *Assume that $N \geq 4$ and Z includes four points in which no three points are collinear. Then it holds that $H^0(S, T_S) = 0, \dim H^1(S, T_S) = 2N - 8$, and $H^2(S, T_S) = 0$.*

PROOF. As it follows from Euler's short exact sequence that $H^0(\mathbb{P}^2, T_{\mathbb{P}^2}) \cong \mathbb{C}^8$ and $H^q(\mathbb{P}^2, T_{\mathbb{P}^2}) = 0$ $(q > 0)$, one can deduce from the exact sequence (1) that $H^2(S, T_S) = 0$ and $\dim H^1(S, T_S) = \dim H^0(S, T_S) + 2N - 8$ (note that here we use the vanishing $R^q \pi_* T_S = 0$ for each $q > 0$ to see $H^q(\mathbb{P}^2, T_{\mathbb{P}^2}) \cong H^q(S, T_S)$). Again by the exact sequence (1) , it is sufficient for proving $H^0(S, T_S) = 0$ to show the restriction $H^0(\mathbb{P}^2, T_{\mathbb{P}^2}) \to H^0(\mathbb{P}^2, j_* N_{Z/\mathbb{P}^2})$ is injective, which can be shown by a simple computation when Z includes four points in which no three points are collinear. $\square$

2.3 Arnol'd's Theorem on a neighborhood of an elliptic curve

A flat line bundle L on an elliptic curve C is said to satisfy the Diophantine condition if $-\log d(\mathbb{I}_C, L^n) = O(\log n)$ as $n \to \infty$, where d is an invariant distance of $\mathrm{Pic}^0(C)$ and $\mathbb{I}_C$ is the holomorphically trivial line bundle on C (see also [U, §4.1]). This condition is independent of the choice of an invariant distance d. Note that the set of all elements of $\mathrm{Pic}^0(C)$ which satisfy the Diophantine condition is a subset of $\mathrm{Pic}^0(C)$ with full Lebesgue

measure, despite the fact that this set is the union of a countable number of nowhere dense Euclidean closed subsets of $\mathrm{Pic}^0(C)$ (see also [A, Proposition 4.3.1]).

In §3, we use the following Arnol'd's theorem for constructing K3 surfaces.

THEOREM 2.3 (=[A, Theorem 4.3.1]). *Let S be a non-singular complex surface and $C \subset S$ be a holomorphically embedded elliptic curve. Assume that $N_{C/S}$ satisfies the Diophantine condition. Then there exists a tubular neighborhood W of C in S which is isomorphic to a neighborhood of the zero section in $N_{C/S}$.*

When the normal bundle $N_{C/S}$ does not satisfy the Diophantine condition, C does not necessarily admit such a neighborhood W of C in S as in Theorem 2.3. Indeed, it follows from Ueda's classification [U, §5] that Ueda's obstruction class $u_n(C,S) \in H^1(C, N_{C/S}^{-n})$ needs to vanish for all $n \geq 1$ if there exists such a neighborhood W as in Theorem 2.3. Ueda also showed the existence of an example of (C,S) with no such a neighborhood W for which all the obstruction classes $u_n(C,S)$'s vanish [U, §5.4]. In this context, the following is one of the most interesting questions.

QUESTION 2.4. Fix a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and take general nine points $Z := \{p_1, p_2, \dots, p_9\} \subset C_0$. Let $S := \mathrm{Bl}_Z\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at Z and $C \subset S$ be the strict transform of C_0. Is there a nine points configuration Z such that $N_{C/S}$ is non-torsion and C does not admit a tubular neighborhood W of C in S which is isomorphic to a neighborhood of the zero section in $N_{C/S}$?

Note that, in this example, all the obstruction classes $u_n(C,S)$'s vanish for any nine points configuration Z (see [N, §6]). See also [B] for this example.

3 Construction of X and Proof of Theorem 1.3

Fix a smooth elliptic curve $C_0 \subset \mathbb{P}^2$ and take general nine points $Z := \{p_1, p_2, \dots, p_9\} \subset C_0$. In what follows, we always assume that Z is sufficiently general so that Z includes four points in which no three points are collinear, and that the line bundle $L_0 := \mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-p_1 - p_2 - \cdots - p_9) \in \mathrm{Pic}^0(C_0)$ satisfies the Diophantine condition. Note that such a nine points configuration actually exists, because almost every $Z \in (\mathbb{P}^2)^9$ satisfies these conditions in the sense of Lebesgue measure (see also §2.3). Let $S := \mathrm{Bl}_Z\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at Z and $C \subset S$ be the strict transform of C_0. Fix another smooth elliptic curve $C_0' \subset \mathbb{P}^2$ and nine points $Z' = \{p_1', p_2', \dots, p_9'\} \subset C_0'$ such that there exists an isomorphism $g\colon C_0' \cong C_0$ and that $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0'} \otimes \mathcal{O}_{C_0'}(-p_1' - p_2' - \cdots - p_9')$ is isomorphic to L_0^{-1} via g. Let $S' := \mathrm{Bl}_{Z'}\mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ at Z' and $C' \subset S'$ be the strict transform of C_0'.

Now we have two models (S,C) and (S',C') as we described in §1. These satisfy the assumption in Theorem 1.3.

Proof of Theorem 1.3

As $N_{C/S}(= L_0)$ satisfies the Diophantine condition, it follows from Theorem 2.3 that there exists a tubular neighborhood W of C such that W is biholomorphic to a neighborhood of the zero section in $N_{C/S}$. Therefore, by shrinking W and considering the pull-back of an open covering $\{U_j\}$ of C by the projection $W \to C$, one can take an open covering $\{W_j\}$ of W and a coordinate system (z_j, w_j) of each W_j which satisfies the following four conditions: (i) W_j is biholomorphic to $U_j \times \Delta_R$, where $\Delta_R := \{w \in \mathbb{C} \mid |w| < R\}$, (ii) W_{jk} is biholomorphic to $U_{jk} \times \Delta_R$, where $W_{jk} := W_j \cap W_k$ and $U_{jk} := U_j \cap U_k$, (iii) z_j can be regarded as a coordinate of U_j and w_j can be regarded as a coordinate of Δ_R, and (iv) $(z_k, w_k) = (z_j + A_{kj}, t_{kj} \cdot w_j)$ holds on W_{jk}, where $A_{kj} \in \mathbb{C}$ and $t_{kj} \in U(1) := \{t \in \mathbb{C} \mid |t| = 1\}$. Note that one can use any positive number for the constant $R > 0$ by rescaling w_j's. In what follows, we always assume that $R > 1$.

Denote by U'_j the subset of C' defined by $g^{-1}(U_j)$ (here we are regarding g as a morphism from C' to C via the natural isomorphisms $C_0 \cong C$ and $C'_0 \cong C'$). Then $\{U'_j\}$ is an open covering of C'. We use the function $z'_j := z_j \circ g$ as a coordinate of U'_j for each j. Again by Theorem 2.3, one can take a tubular neighborhood W' of C' in S', an open covering $\{W'_j\}$ of W', and a coordinate system (z'_j, w'_j) of each W_j which satisfies the following four conditions: $(i)'$ W'_j is biholomorphic to $U'_j \times \Delta_{R'}$ $(R' > 1)$, $(ii)'$ W'_{jk} is biholomorphic to $U'_{jk} \times \Delta_{R'}$, $(iii)'$ z'_j is the coordinate of U'_j as above and w'_j can be regarded as a coordinate of $\Delta_{R'}$, and $(iv)'$ $(z'_k, w'_k) = (z'_j + A_{kj}, t_{kj}^{-1} \cdot w_j)$ holds on W'_{jk}, where A_{kj} and t_{kj} are the same constants as in the above condition (iv).

Set $W^* := \bigcup_j W_j^*$, where $W_j^* := \{(z_j, w_j) \in W_j \mid 1/R' < |w_j| < R\}$ and regard it as a subset of S. Denote by M the connected component of $S \setminus W^*$ which does not include C. Define a holomorhic map $f \colon W^* \to W'$ by

$$f|_{W_j^*} \colon (z_j, w_j) \mapsto (z'_j(z_j, w_j), w'_j(z_j, w_j)) := \left(g^{-1}(z_j), w_j^{-1}\right) \in W'_j$$

on each W_j^* and regard W^* also as a subset of S' via this map f. Denote by M' the connected component of $S' \setminus W^* (:= S' \setminus f(W^*))$ which does not include C'. Then we can patch M and M' by using the map f to define a compact complex surface X. In what follows, we regard M, M', and W^* as open subsets of X. Note that the open subset $V := W^* \subset X$ satisfies the conditions as in §1.

Proposition 3.1. *X is a K3 surface with a global holomorhic 2-form σ with*

$$\sigma|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j}$$

on each $W_j^* \subset X$.

Proof. As it easily follows from Mayer–Vietoris sequence associated to the open covering $\{M, W\}$ of S that $H_1(M, \mathbb{Z}) = 0$. Again by Mayer–Vietoris sequence associated to the open covering $\{M, M'\}$ of X, we have that $H_1(X, \mathbb{Z}) = 0$. Therefore it is sufficient to show the existence of a nowhere vanishing global holomorhic 2-form σ with $\sigma|_{W_j^*} = dz_j \wedge \frac{dw_j}{w_j}$ on each $W_j^* \subset X$. As it holds that $K_S = -C$, S admits a global meromorphic 2-form η with no zero and with poles only along C. Define a nowhere vanishing holomorphic function F_j on W_j^* by

$$F_j := \frac{\eta}{dz_j \wedge \frac{dw_j}{w_j}}.$$

Then the functions $\{(W_j^*, F_j)\}$ glue up to define a holomorphic function $F\colon W^* \to \mathbb{C}$. By the following Lemma 3.2, we have that F is a constant function $F \equiv A \in \mathbb{C}^*$. Therefore it follows that $\eta|_{W_j^*} = A \cdot dz_j \wedge \frac{dw_j}{w_j}$. Similarly, we obtain that a meromorphic 2-form η' with poles only along C satisfies $\eta'|_{W_j^*} = A' \cdot dz_j \wedge \frac{dw'_j}{w'_j}$ on each W'_j for some constant $A' \in \mathbb{C}^*$. As $f^*\frac{dw'_j}{w'_j} = -\frac{dw_j}{w_j}$, we have that $-A^{-1} \cdot \eta|_M$ and $(A')^{-1} \cdot \eta'|_{M'}$ glue up to define a nowhere vanishing 2-form σ on X, which shows the theorem. □

Lemma 3.2. $H^0(W^*, \mathcal{O}_{W^*}) = \mathbb{C}$.

Proof. Let $F\colon W^* \to \mathbb{C}$ be a holomorphic function. Take a real number r with $1/R' < r < R$ and a point $x_r \in M_r := \bigcup_j \{x \in W_j^* \mid |x| = r\}$ which attains the maximum value $\max_{x \in M_r} |F(x)|$. Denote by L the leaf of the Levi-flat M_r with $x_r \in L$. By the maximum modulus principle for $F|_L$ and the density of $L \subset M$, it follows that $F|_{M_r} \equiv A$ for some constant $A \in \mathbb{C}$. As $\{x \in W^* \mid F(x) = A\}$ is an analytic subvariety of W^* which includes a real three dimensional submanifold M_r, we have that $\{x \in W^* \mid F(x) = A\} = W^*$. □

Theorem 1.3 follows from Proposition 3.1. □

Remark 3.3. As it is easily seen from the construction above, we can replace the condition "that C and C' are biholomorphic and $N_{C/S} \cong N_{C'/S'}^{-1}$" in Theorem 1.3 with the following looser condition: there exists a biholomorphism $N_{C/S}^* \cong N_{C'/S'}^*$ which maps the connected component of the boundary of $N_{C/S}^*$ corresponding to the zero section of $N_{C/S}$ to the connected component of the boundary of $N_{C'/S'}^*$ corresponding to the boundary of $N_{C'/S'}$, where $N_{C/S}^*$ and $N_{C'/S'}^*$ are the complement of the zero section of $N_{C/S}$ and $N_{C'/S'}$, respectively.

For the open subset $W^* \subset X$, we have the following:

Proposition 3.4. *Denote by $\widehat{W}_{r,r'}^*$ the subset $\{x \in L_0 \mid 1/r' < |x|_h < r\}$ for positive numbers $r > 1$ and $r' > 1$, where h is a fiber metric on L_0 with zero curvature. Denote by i_0*

the natural isomorphism $\widehat{W}^*_{R,R'} \to W^*(\subset X)$*. Then it holds that* $\sup\{r \geq R \mid \text{there exists a holomorphic embedding } i_{r,R'} \colon \widehat{W}^*_{r,R'} \to X \text{ with } i_{r,R'}|_{\widehat{W}^*_{R,R'}} = i_0\} < \infty$ *and* $\sup\{r' \geq R' \mid \text{there exists a holomorphic embedding } i_{R,r'} \colon \widehat{W}^*_{R,r'} \to X \text{ with } i_{R,r'}|_{\widehat{W}^*_{R,R'}} = i_0\} < \infty$.

Proof. Take $r \geq R$ and $r' \geq R'$ such that there exists a holomorphic embedding $i_{r,r'} \colon \widehat{W}^*_{r,r'} \to X$ with $i_{r,r'}|_{\widehat{W}^*_{R,R'}} = i_0$. Then we can calculate to obtain that

$$\int_X \sigma \wedge \overline{\sigma} \geq \int_{i_{r,r'}(\widehat{W}^*_{r,r'})} \sigma \wedge \overline{\sigma} = 4\pi \cdot \left(\int_C \sqrt{-1}\eta_C \wedge \overline{\eta_C} \right) \cdot \log(rr'),$$

where σ is as in Proposition 3.1 and η_C is the holomorphic 1-form on C defined by $\eta_C|_{U_j} = dz_j$ on each U_j. Therefore we have an inequality

$$\log r + \log r' \leq \frac{\int_X \sigma \wedge \overline{\sigma}}{4\pi \int_C \sqrt{-1}\eta_C \wedge \overline{\eta_C}},$$

which proves the proposition. □

Remark 3.5. By the same argument as in the proof of Proposition 3.4, we can show the following statement on tubular neighborhoods of C in S: Denoting by $\widehat{W}_r$ the subset $\{x \in L_0 \mid |x|_h < r\}$ and by i_R the natural isomorphism $\widehat{W}_R \to W \subset S$, it holds that $\sup\{r \geq R \mid \text{there exists a holomorphic embedding } i_r \colon \widehat{W}_r \to S \text{ with } i_r|_{\widehat{W}_R} = i_R\} < \infty$.

4 Deformation of X

Let $C_0, C_0', L_0, Z = \{p_1, p_2, \dots, p_r\}, Z' = \{p_1', p_2', \dots, p_r'\}$, and X be those in §3. The construction of X in the previous section has some degrees of freedom: on the points configurations Z and Z', and on the patching functions, even after fixing C_0, C_0' and L_0. In this section, we investigate some of the deformation families constructed by considering such degrees of freedom.

4.1 Deformation families constructed by changing the patching functions

Let $S, S', C, C', M, M', W, W', R, R', g$, and W^* be those in §3.

4.1.1 A deformation family corresponding to the change of the patching function $w_j'(z_j, w_j)$

Here we fix the isomorphism g. Denote by $A_{R'}$ the annulus $\{t \in \mathbb{C} \mid 1 < |t| < R'\}$ and by $\mathcal{W}^*$ the union $\bigcup_j \mathcal{W}_j^* \subset W \times A_{R'}$, where $\mathcal{W}_j^* := \{(z_j, w_j, t) \in W_j \times A_{R'} \mid |t|^{-1} < |w_j| < R\}$.

Denote by $\mathcal{M}$ the connected component of $S \times A_{R'} \setminus \mathcal{W}^*$ which does not include $C \times A_{R'}$. Define a holomorphic function $F \colon \mathcal{W}^* \to \mathcal{M}' := M' \times A_{R'}$ by

$$F|_{\mathcal{W}_j^*} \colon (z_j, w_j, t) \mapsto (z_j'(z_j, w_j, t), w_j'(z_j, w_j, t), t) := \left(g^{-1}(z_j), R' \cdot (t \cdot w_j)^{-1}, t\right)$$

on each $\mathcal{W}_j^*$, and regard $\mathcal{W}^*$ also as a subset of $\mathcal{M}'$ via this map F. Then we can patch $\mathcal{M}$ and $\mathcal{M}'$ by using the map F to define a complex manifold $\mathcal{X}$. In what follows, we regard $\mathcal{M}, \mathcal{M}'$, and $\mathcal{W}^*$ as open subsets of $\mathcal{X}$. The second projections $\mathrm{Pr}_2 \colon \mathcal{M} \to A_{R'}$ and $\mathrm{Pr}_2 \colon \mathcal{M}' \to A_{R'}$ glue up to define a proper holomorphic submersion $\pi \colon \mathcal{X} \to A_{R'}$. By Proposition 3.1, we have that each fiber $X_t := \pi^{-1}(t)$ is a K3 surface. In what follows, we fix a base point $t_0 \in A_{R'}$.

PROPOSITION 4.1. *The Kodaira–Spencer map $\rho_{\mathrm{KS},\pi} \colon T_{A_{R'},t_0} \to H^1(X_{t_0}, T_{X_{t_0}})$ of the deformation family $\pi \colon \mathcal{X} \to A_{R'}$ is injective.*

Denote by θ_1 the holormophic vector field on $W_{t_0}^* := X_{t_0} \cap \mathcal{W}^*$ defined by $\{(X_{t_0} \cap \mathcal{W}_j^*, w_j \frac{\partial}{\partial w_j})\}$, and by θ_2 the holomorphic vector field defined by $\{(X_{t_0} \cap \mathcal{W}_j^*, \frac{\partial}{\partial z_j})\}$. Proposition 4.1 follows from the following two lemmata.

LEMMA 4.2. *The image of $\frac{\partial}{\partial t}\big|_{t_0}$ by the Kodaira–Spencer map $\rho_{\mathrm{KS},\pi}$ is given by $[\{(M_{t_0} \cap M_{t_0}', t_0^{-1} \cdot \theta_1)\}] \in \check{H}^1(\{M_{t_0}, M_{t_0}'\}, T_{X_{t_0}})$ as a Čech cohomology class, where $M_{t_0} := X_{t_0} \cap \mathcal{M}$ and $M_{t_0}' := X_{t_0} \cap \mathcal{M}'$.*

PROOF. Lemma directly follows from the computation

$$\frac{\partial w_j'(z_j, w_j, t)}{\partial t} \frac{\partial}{\partial w_j'} = \left(\frac{\partial}{\partial t} \frac{R'}{t} \cdot w_j^{-1}\right) \cdot \frac{\partial}{\partial w_j'} = -t^{-1} w_j' \frac{\partial}{\partial w_j'} = t^{-1} \cdot w_j \frac{\partial}{\partial w_j}.$$

□

LEMMA 4.3. *The coboundary map $H^0(W_{t_0}^*, T_{W_{t_0}^*}) \to H^1(X_{t_0}, T_{X_{t_0}})$ which appears in the Mayer–Vietoris sequence corresponding to the covering $\{M_{t_0}, M_{t_0}'\}$ of X_{t_0} is injective.*

See [I, II 5.10] for example for the Mayer–Vietoris sequence for an open covering.

PROOF. By considering the Mayer–Vietoris sequence, it is sufficient to show that $H^0(M_{t_0}, T_{M_{t_0}}) = 0$ and $H^0(M_{t_0}', T_{M_{t_0}'}) = 0$ hold. Take an element $\xi \in H^0(M_{t_0}, T_{M_{t_0}})$. As $\xi|_{W_{t_0}^*} \in H^0(W_{t_0}^*, T_{W_{t_0}^*})$ and $T_{W_{t_0}^*} = \theta_1 \cdot \mathbb{I}_{W_{t_0}^*} \oplus \theta_2 \cdot \mathbb{I}_{W_{t_0}^*}$, it follows from Lemma 3.2 that there exists an element $(a_1, a_2) \in \mathbb{C}^2$ such that $\xi|_{W_{t_0}^*} = a_1 \cdot \theta_1 + a_2 \cdot \theta_2$. As it is clear that both θ_1 and θ_2 can be extended to holomorphic vector fields on $W_{t_0} := W \times \{t_0\}$, it follows that there exists $\zeta \in H^0(W_{t_0}, T_{W_{t_0}})$ such that $\zeta|_{W_{t_0}^*} = \xi|_{W_{t_0}^*}$. By gluing ζ and ξ, we obtain a global vector field $\widetilde{\xi} \in H^0(S, T_S)$. By Lemma 2.2, we have that $\widetilde{\xi} = 0$. Therefore we have that $\xi = 0$, which shows that $H^0(M_{t_0}, T_{M_{t_0}}) = 0$. Similarly we have $H^0(M_{t_0}', T_{M_{t_0}'}) = 0$. □

Proof of Proposition 4.1. As the map $\check{H}^1(\{M_{t_0}, M_{t_0}'\}, T_{X_{t_0}}) \to \lim_{\mathcal{U} \to} \check{H}^1(\mathcal{U}, T_{X_{t_0}}) = H^1(X_{t_0}, T_{X_{t_0}})$ is injective, Proposition 4.1 follows from Lemma 4.2 and Lemma 4.3. □

4.1.2 A deformation family corresponding to the change of the patching function $z'_j(z_j, w_j)$

Denote by Δ the disc $\{t \in \mathbb{C} \mid |t| < 1\}$. Let $\widetilde{g}\colon \mathbb{C} \to \mathbb{C}$ be the isomorphism such that $p_C \circ \widetilde{g} = g \circ p_{C'}$ holds, where $p_C\colon \mathbb{C} \to C$ and $p_{C'}\colon \mathbb{C} \to C'$ are the universal covers. Without loss of generality, we may assume that $\frac{\partial}{\partial z}\widetilde{g}(z) \equiv 1$. Denote by $\widetilde{g}_t$ the automorphism of $\mathbb{C}$ defined by $\widetilde{g}_t(z) := \widetilde{g}(z) - t$ and by $g_t\colon C' \to C$ the isomorphism induced by $\widetilde{g}_t$ for each $t \in \Delta$. Let $\mathcal{M} := M \times \Delta$, $\mathcal{M}' := M' \times \Delta$, $\mathcal{W}^*_j := W^*_j \times \Delta$ for each j, and $\mathcal{W}^* := W^* \times \Delta$. Let $\mathcal{W}'_j$ be the subset of $\mathcal{W}' := W' \times \Delta$ defined by $\mathcal{W}'_j := \{(z'_j, w'_j, t) \mid z'_j \in g_t^{-1}(U_j), |w'_j| < R', t \in \Delta\}$. Define a holomorphic function $F\colon \mathcal{W}^* \to \mathcal{M}'$ by

$$F|_{\mathcal{W}^*_j}\colon (z_j, w_j, t) \mapsto (z'_j(z_j, w_j, t), w'_j(z_j, w_j, t), t) := \left(g_t^{-1}(z_j), w_j^{-1}, t\right) \in \mathcal{W}'_j$$

on each $\mathcal{W}^*_j$. Regard $\mathcal{W}^*$ also as a subset of $\mathcal{M}'$ via this map F. Then we can patch $\mathcal{M}$ and $\mathcal{M}'$ by using the map F to define a complex manifold $\mathcal{X}$. We regard $\mathcal{M}, \mathcal{M}'$, and $\mathcal{W}^*$ as open subsets of $\mathcal{X}$. The second projections $\mathrm{Pr}_2\colon \mathcal{M} \to \Delta$ and $\mathrm{Pr}_2\colon \mathcal{M}' \to \Delta$ glue up to define a proper holomorphic submersion $\pi\colon \mathcal{X} \to \Delta$. By Proposition 3.1, we have that each fiber $X_t := \pi^{-1}(t)$ is a K3 surface.

Proposition 4.4. *The Kodaira–Spencer map $\rho_{\mathrm{KS},\pi}\colon T_{\Delta,0} \to H^1(X_0, T_{X_0})$ of the deformation family $\pi\colon \mathcal{X} \to \Delta$ is injective.*

As in the previous subsection, we denote by θ_2 the holomorphic vector field defined by $\{(X_0 \cap \mathcal{W}^*_j, \frac{\partial}{\partial z_j})\}$. Proposition 4.4 follows from the following:

Lemma 4.5. *The image of $\frac{\partial}{\partial t}\big|_{t=0}$ by the Kodaira–Spencer map is given by $[\{(M_0 \cap M'_0, \theta_2)\}] \in \check{H}^1(\{M_0, M'_0\}, T_{X_0})$ as a Čech cohomology class, where $M_0 := X_0 \cap \mathcal{M}$ and $M'_0 := X_0 \cap \mathcal{M}'$.*

Proof. Lemma directly follows from the computation

$$\frac{\partial z'_j(z_j, w_j, t)}{\partial t}\frac{\partial}{\partial z'_j} = \left(\frac{\partial}{\partial t}(g^{-1}(z_j) + t)\right) \cdot \frac{\partial}{\partial z'_j} = \frac{\partial}{\partial z'_j} = \frac{\partial}{\partial z'_j} g_t(z'_j) \cdot \frac{\partial}{\partial z_j} = \frac{\partial}{\partial z_j}.$$

□

Proof of Proposition 4.4. As in the proof of Proposition 4.1, Proposition 4.4 follows from Lemma 4.3 and Lemma 4.5. □

4.2 A deformation family constructed by changing the nine points configurations

Let $C_0, C_0', L_0, Z = \{p_1, p_2, \dots, p_r\}, Z' = \{p_1', p_2', \dots, p_r'\}, S, S', C, C', M, M', W, W', R, R', g$, and W^* be those in §3. In this subsection, we construct a deformation family corresponding to the change of the nine points configurations Z (and Z') by fixing C_0, C_0', L_0 and g. For simplicity, we also fix Z' and consider only the change of Z here.

Fix a sufficiently small open neighborhood U_ν of p_ν in C_0 for each $\nu = 1, 2, \dots, 8$ and denote by T the product $U_1 \times U_2 \times \cdots \times U_8$. In what follows, we regard $t_0 := (p_1, p_2, \dots, p_8)$ as a base point of T. For each $t = (q_1, q_2, \dots, q_8) \in T$, we define $q(t) \in C_0$ by $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-q_1 - q_2 - \cdots - q_8 - q(t)) \cong L_0$. Let $\pi\colon \mathcal{S} \to T$ be a proper holomorphic submersion from a 10-dimensional complex manifold $\mathcal{S}$ to T such that each fiber $S_t := \pi^{-1}(t)$ is isomorphic to the blow-up of $\mathbb{P}^2$ at nine points $q_1, q_2, \dots, q_8$, and $q(t)$ for each $t = (q_1, q_2, \dots, q_8) \in T$. Such $\mathcal{S}$ can be constructed as the blow-up of $\mathbb{P}^2 \times T$ along some submanifolds. Let $\mathcal{C} \subset \mathcal{S}$ be the strict transform of $C_0 \times T$. Note that $\pi|_{\mathcal{C}} = \mathrm{Pr}_2\colon C_0 \times T \to T$ holds via the natural isomorphism between $\mathcal{C}$ and $C_0 \times T$. Denote by C_t the intersection $S_t \cap \mathcal{C}$ for each $t \in T$. Then it follows from the construction that $N_{\mathcal{C}/\mathcal{S}}|_{C_t} = N_{C_t/S_t} \cong L_0$ for each t. Therefore, by regarding $N_{\mathcal{C}/\mathcal{S}}$ as a holomorphic line bundle on $C_0 \times T$, it follows from Proprosition 2.1 that there exists a holomorphic map $i\colon T \to \mathrm{Pic}^0(C_0)$ such that $(\mathrm{id}_{C_0} \times i)^*\mathcal{L} \cong N_{\mathcal{C}/\mathcal{S}}$ holds, where $\mathcal{L}$ is the universal line bundle on $C_0 \times \mathrm{Pic}^0(C_0)$. As it clearly holds that $i(t) \equiv L_0 \in \mathrm{Pic}^0(C_0)$, we obtain that $N_{\mathcal{C}/\mathcal{S}} \cong \mathrm{Pr}_1^* L_0$. Then, we can apply the following relative variant of Arnol'd's theorem to our $(\mathcal{C}, \mathcal{S})$.

THEOREM 4.6. *Let $\pi\colon \mathcal{S} \to T$ be a deformation family of complex surfaces over a ball in $\mathbb{C}^n$, and $\mathcal{C} \subset \mathcal{S}$ be a submanifold which is biholomorphic to $C_0 \times T$, where C_0 is an elliptic curve, and satisfies $\pi|_{\mathcal{C}} = \mathrm{Pr}_2$ via this biholomorphism. Assume that $N_{\mathcal{C}/\mathcal{S}} \cong \mathrm{Pr}_1^* L$ holds for some line bundle L on C_0 which satisfies the Diophantine condition. Then, by shrinking T if necessary, there exists a tubular neighborhood $\mathcal{W}$ of $\mathcal{C}$ in $\mathcal{S}$ which is isomorphic to a neighborhood of the zero section in $N_{\mathcal{C}/\mathcal{S}}$.*

See §6.1 for the proof of Theorem 4.6. Take $\mathcal{W} \subset \mathcal{S}$ as in Theorem 4.6. By shrinking $\mathcal{W}$ and considering the pull-back of an open covering $\{U_j \times T\}$ of $\mathcal{C}$ by the projection $\mathcal{W} \to \mathcal{C}$, we can take an open covering $\{\mathcal{W}_j\}$ of $\mathcal{W}$ and a coordinate system (z_j, w_j, t) of each $\mathcal{W}_j$ which satisfies the following four conditions: (I) $\mathcal{W}_j$ is biholomorphic to $U_j \times \Delta_R \times T$ $(R > 1)$, (II) $\mathcal{W}_{jk}$ is biholomorphic to $U_{jk} \times \Delta_R \times T$, (III) $\pi(z_j, w_j, t) = t$ holds on each $\mathcal{W}_j$, and (IV) $(z_k, w_k, t) = (z_j + A_{kj}, t_{kj} \cdot w_j, t)$ holds on $\mathcal{W}_{jk}$, where A_{kj} and t_{kj} are as in the condition (iv) in §3. Denote by $\mathcal{W}^*$ the union of $\mathcal{W}_j^*$'s, where $\mathcal{W}_j^* := \{(z_j, w_j, t) \in \mathcal{W}_j \mid 1/R' < |w_j| < R\}$. Let $\mathcal{M}$ be the connected component of

$\mathcal{S} \setminus \mathcal{W}^*$ which does not contain $\mathcal{C}$. Define a holomorphic function $F\colon \mathcal{W}^* \to \mathcal{S}' := S' \times T$ by $F(z_j, w_j, t) := (g^{-1}(z_j), w_j^{-1}, t) \in \mathcal{W}_j' := W_j' \times T$ on each $\mathcal{W}_j^*$. By regarding $\mathcal{W}^*$ as a subset of $\mathcal{M}' := M' \times T$ via F, we patch $\mathcal{M}$ and $\mathcal{M}'$ to construct a manifold $\mathcal{X}$ just as in the previous section. The maps π and $\pi' := \mathrm{Pr}_2\colon \mathcal{S}' \to T$ glue up to define a proper holomorphic submersion $P\colon \mathcal{X} \to T$. By the same argument as in the proof of Proposition 3.1, we have that each fiber $X_t := P^{-1}(t)$ is a K3 surface.

PROPOSITION 4.7. *If one choose $Z = \{p_1, p_2, \dots, p_9\}$ appropriately, the Kodaira–Spencer map $\rho_{\mathrm{KS},P}\colon T_{T,t_0} \to H^1(X_{t_0}, T_{X_{t_0}})$ is injective.*

PROOF. First, consider an open covering $\{M_0 := S_{t_0} \cap \mathcal{M},\ W_0 := S_{t_0} \cap \mathcal{W}\}$ of S_{t_0}. By Mayer–Vietoris sequence for open coverings [I, II 5.10], we have a long exact sequence

$$H^0(M_0, T_{M_0}) \oplus H^0(W_0, T_{W_0}) \to H^0(W_0^*, T_{W_0^*}) \to H^1(S_{t_0}, T_{S_{s_0}}) \to H^1(M_0, T_{M_0}) \oplus H^1(W_0, T_{W_0}),$$

where $W_0^* := M_0 \cap W_0$. By the arguments as in the proof of Proposition 4.1, it follows that the map $H^0(W_0, T_{W_0}) \to H^0(W_0^*, T_{W_0^*})$ is surjective. Therefore we have that the map $H^1(S_{t_0}, T_{S_{t_0}}) \to H^1(M_0, T_{M_0}) \oplus H^1(W_0, T_{W_0})$ is injective.

By considering a real number $\widetilde{R}$ slightly larger than R, we can take another open neighborhood $\mathcal{V}$ of $\mathcal{C}$ in $\mathcal{S}$ and an open covering $\{\mathcal{V}_j\}$ of $\mathcal{V}$ which satisfies the above conditions $(I), (II), (III)$ and (IV) after replacing R with $\widetilde{R}$. Note that $\mathcal{W} \Subset \mathcal{V}$ and $\mathcal{W}_j \subset \mathcal{V}_j$ for each j. By using sufficiently fine open covering $\{\mathcal{U}_\lambda\}$ of $\mathcal{M} \setminus \mathcal{W}$ and by regarding $\{\mathcal{U}_\lambda\} \cup \{\mathcal{V}_j\}$ as a open covering of $\mathcal{S}$, we can compute the Kodaira-Spencer map $\rho_{\mathrm{KS},\pi}$. Note that the intersection of two of the elements of $\{\mathcal{U}_\lambda\} \cup \{\mathcal{V}_j\}$ is in the form of $\mathcal{V}_{jk}$ if it intersects $\mathcal{W}$. As the fiber coordinates of each $\mathcal{V}_{jk}$ do not depend on the coordinate t of T, it follows by the definition of Kodaira–Spencer map that the composition of $\rho_{\mathrm{KS},\pi}$ and the restriction map $H^1(S_{t_0}, T_{S_{t_0}}) \to H^1(W_0, T_{W_0})$ is the zero map. Therefore, it follows from Lemma 4.9 below that the composition $r_1 \circ \rho_{\mathrm{KS},\pi}$ is injective, where $r_1\colon H^1(S_{t_0}, T_{X_{t_0}}) \to H^1(M_0, T_{M_0})$ is the restriction map.

Next, let us consider an open covering $\{\mathcal{U}_\lambda\} \cup \{\mathcal{V}_j^*\} \cup \{\mathcal{M}_\lambda'\}$ of $\mathcal{X}$, where $\mathcal{V}_j^* := \{(z_j, w_j, t) \in \mathcal{V}_j \mid 1/R' < |w_j| < \widetilde{R}\}$ and $\{\mathcal{M}_\lambda'\}$ is a sufficiently fine open covering of M'. Then it follows by the definition of Kodaira–Spencer map that the two morphisms $r_1 \circ \rho_{\mathrm{KS},\pi}$ and $r_2 \circ \rho_{\mathrm{KS},P}$ coincide with each other, where $r_2\colon H^1(X_{t_0}, T_{X_{t_0}}) \to H^1(M_0, T_{M_0})$ is the restriction map. Therefore we have that $\rho_{\mathrm{KS},P}$ is also injective, which proves the proposition. $\square$

REMARK 4.8. By the arguments as in the proof of Proposition 4.7, it also follows by the definition of Kodaira–Spencer map that the composition of $\rho_{\mathrm{KS},P}$ and the restriction map $H^1(X_{t_0}, T_{X_{t_0}}) \to H^1(M_0', T_{M_0'})$ is the zero map.

Lemma 4.9. *If one choose $Z = \{p_1, p_2, \dots, p_9\}$ appropriately, the Kodaira–Spencer map $\rho_{\mathrm{KS},\pi}\colon T_{T,t_0} \to H^1(S_{t_0}, T_{S_{t_0}})$ is injective.*

Proof. By Kodaira and Kuranishi's theorem, it follows from Lemma 2.2 that S_{t_0} admits a Kuranishi family $\widetilde{\pi}\colon \widetilde{\mathcal{S}} \to \widetilde{T}$, where $\widetilde{T}$ is a manifold of dimension 10 and $\widetilde{\pi}$ is complete and universal on each point of $\widetilde{T}$ (See [V, Theorem 3.6.3.1], [H, Theorem 2.5], for example). Denote by $\widetilde{S}_t$ the fiber $\widetilde{\pi}^{-1}(t)$ for each $t \in \widetilde{T}$. We fix a base point $0 \in \widetilde{T}$ with $\widetilde{S}_0 = S_{t_0}$. By shrinking T if necessary, there uniquely exists a holomorphic map $f\colon T \to \widetilde{T}$ with $f(t_0) = 0$ and $f^*\widetilde{\mathcal{S}} = \mathcal{S}$. Then it holds that $\rho_{\mathrm{KS},\pi} = \rho_{\mathrm{KS},\widetilde{\pi}} \circ D_{t_0} f$, where $D_{t_0} f\colon T_{T,t_0} \to T_{\widetilde{T},0}$ is the differential of f at t_0 and $\rho_{\mathrm{KS},\widetilde{\pi}}\colon T_{\widetilde{T},0} \to H^1(\widetilde{S}_0, T_{\widetilde{S}_0}) = H^1(S_{t_0}, T_{S_{t_0}})$ is the Kodaira–Spencer map of $\widetilde{\pi}\colon \widetilde{\mathcal{S}} \to \widetilde{T}$. As $\rho_{\mathrm{KS},\widetilde{\pi}}$ is an isomorphism (see [V, Theorem 3.6.3.1] for example), it is sufficient to show the injectivity of $D_{t_0} f$.

It follows from the following Lemma 4.10 that we may assume that $f\colon T \to \widetilde{T}$ is injective. Therefore, by proper mapping theorem, we can regard T as an analytic subvariety of $\widetilde{T}$ via f. Thus we may assume that t_0 is a smooth point of the subvariety $T \subset \widetilde{T}$ by moving t_0 slightly if necessary, which proves the injectivity of $D_{t_0} f$. □

Lemma 4.10. *If one choose $Z = \{p_1, p_2, \dots, p_9\}$ appropriately, the morphism $f\colon T \to \widetilde{T}$ as in the proof of Lemma 4.9 is injective on a neighborhood of t_0.*

Proof. Take two points $t_1, t_2 \in T$ and assume that $f(t_1) = f(t_2)$ holds. Denote by $Z_\nu = \{p_1^{(\nu)}, p_2^{(\nu)}, \dots, p_9^{(\nu)}\}$ the nine points configurations corresponding to t_ν for $\nu = 1, 2$. Then, as $S_{t_1} \cong S_{t_2}$, there exists a holomorphic automorphism $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2$ with $\varphi(Z_1) = Z_2$. From this observation, we obtain that it is sufficient to show the existence of $Z = \{p_1, p_2, \dots, p_9\}$ (which includes four points in which no three points are collinear) and a neighborhood U_λ of p_λ in C_0 for each $\lambda = 1, 2, \dots, 9$ which have the following property: a holomorphic automorphism $\varphi\colon \mathbb{P}^2 \to \mathbb{P}^2$ is the identity if there exists nine points configurations $Z_1, Z_2 \subset \bigcup_{\mu=1}^9 U_\mu$ such that $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-\sum_{\mu=1}^9 p_\mu^{(\nu)}) \cong L_0$ holds for each $\nu = 1, 2$ and $\varphi(Z_1) = Z_2$.

Let Z_1, Z_2, and φ be as above. As L_0 is non-torsion, we have that C_0 is only the elliptic curve in $\mathbb{P}^2$ passing through Z_ν for each $\nu = 1, 2$. Therefore it follows that $\varphi(C_0) = C_0$. Let $\{\psi_\lambda\}_{\lambda=1}^N$ be the set of all holomorphic automorphisms of $\mathbb{P}^2$ which have C_0 as a invariant subset. Note that, as such a map ψ_λ must map an inflection point to an inflection point. Thus, by considering four inflection points of C_0 in which no three points are collinear, we have that $N \leq 9 \times 8 \times 7 \times 6 < \infty$. Let M_λ be a 3×3 matrix corresponding to ψ_λ. Denote by $\{e_{\lambda,\mu}\}_\mu$ the set of all eigenvalues of M_λ. Then the set of all fixed points of ψ_λ can be written as

$$\bigcup_\mu \left[\ker \left(M_\lambda - \begin{pmatrix} e_{\lambda,\mu} & 0 & 0 \\ 0 & e_{\lambda,\mu} & 0 \\ 0 & 0 & e_{\lambda,\mu} \end{pmatrix} \right) \right] \subset \mathbb{P}^2,$$

which is the union of finite linear subspaces of $\mathbb{P}^2$ of dimension less than 2 if ψ_λ is not the identity. Therefore we have that the set $Q := \bigcup_{\psi_\lambda \neq \mathrm{id}_{\mathbb{P}^2}} \{x \in C_0 \mid \psi_\lambda(x) = x\}$ is a finite set.

Take p_1 from $C_0 \backslash Q$. Then, by taking a sufficiently small neighborhood U_1 of p_1, we may assume that $\psi_\lambda(U_1) \cap U_1 = \emptyset$ for each $\psi_\lambda \neq \mathrm{id}_{\mathbb{P}^2}$. Take $p_2, p_3, \dots, p_9$ from $C_0 \setminus \bigcup_\lambda \psi_\lambda(U_1)$. Then we may assume that $\bigcup_{\nu=2}^9 U_\nu \cap \psi_\lambda(U_1) = \emptyset$ for each ψ_λ by taking sufficiently small $U_2, U_3, \dots, U_9$. Then it holds that, for any $q_1 \in U_1$, $\psi_\lambda = \mathrm{id}_{\mathbb{P}^2}$ holds if $\psi_\lambda(q_1) \in \bigcup_{\nu=1}^9 U_\nu$. Finally we check the property above by using this choice of p_ν's and U_ν's. Let $\varphi \colon \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic automorphism and $Z_1, Z_2 \subset \bigcup_{\mu=1}^9 U_\mu$ be nine points configurations such that $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-\sum_{\mu=1}^9 p_\mu^{(\nu)}) \cong L_0$ holds for each $\nu = 1, 2$ and $\psi(Z_1) = Z_2$. As we have already seen, this φ must have C_0 as an invariant set. Thus, $\varphi = \psi_\lambda$ for some λ. As $\psi(p_1^{(1)}) \in Z_2 \subset \bigcup_{\mu=1}^9 U_\mu$, we have that φ is the identity. $\square$

Remark 4.11. Each of the condition of t_0 we considered in the proof of Proposition 4.7 is open condition in T. Therefore we can see that the Kodaira–Spencer map $\rho_{\mathrm{KS},P} \colon T_{T,t} \to H^1(X_t, T_{X_t})$ is injective for any point $t \in T$ by shrinking T if necessary.

4.3 A deformation family corresponding to the change of both the patching function and the nine points configurations

Fix C_0, C_0', and L_0. Take $Z = \{p_1, p_2, \dots, p_9\}$ and $Z' = \{p_1', p_2', \dots, p_9'\}$ appropriately so that Proposition 4.7 holds. In this subsection, we use an 18-dimensional complex manifold $T := A_{R'} \times \Delta \times U_1 \times U_2 \times \cdots \times U_8 \times U_1' \times U_2' \times \cdots \times U_8'$ as a parameter space, where Δ, $A_{R'}$ and U_ν are those in §4.1 and §4.2, and U_ν' is a neighborhood of p_ν' in C_0'. By combining the constructions of the deformation families as in §4.1 and §4.2, one can naturally construct a deformation family $\pi \colon \mathcal{X} \to T$ and the subsets $\mathcal{M}, \mathcal{M}' \subset \mathcal{X}$ such that the following conditions hold: for each $t = (t_1, t_2, q_1, q_2, \dots, q_8, q_1', q_2', \cdots, q_8') \in T$, M_t and M_t' are subsets of the blow-up of $\mathbb{P}^2$ at $\{q_1, q_2, \dots, q_8, q_9\}$ and $\{q_1', q_2', \cdots, q_8', q_9'\}$ respectively, and X_t is a K3 surface obtained by patching M_t and M_t' by identifying the point $(z_j, w_j) \in M_t$ with the point $(z_j'(z_j, w_j), w_j'(z_j, w_j)) := \left(g_{t_2}^{-1}(z_j), R' \cdot (t_1 \cdot w_j)^{-1}\right) \in M_t'$, where $X_t := \pi^{-1}(t)$ is the fiber, $M_t := \mathcal{M} \cap X_t$, $M_t' := \mathcal{M}' \cap X_t$, $q_9 \in C_0$ and $q_9' \in C_0'$ are the points such that $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0} \otimes \mathcal{O}_{C_0}(-q_1 - q_2 - \cdots - q_8 - q_9) \cong L_0$ and $\mathcal{O}_{\mathbb{P}^2}(3)|_{C_0'} \otimes \mathcal{O}_{C_0'}(-q_1' - q_2' - \cdots - q_8' - q_9') \cong L_0^{-1}$, and (z_j, w_j)'s and (z_j', w_j')'s are the coordinates near the boundary of M_t and M_t', respectively, with $\frac{1}{|t_1|} < |w_j| < R, \frac{1}{R'} < |w_j'| < R'$, $(z_k, w_k) = (z_j + A_{kj}, t_{kj} \cdot w_j)$, and $(z_k', w_k') = (z_j' + A_{kj}, t_{kj}^{-1} \cdot w_j')$.

Proposition 4.12. *By shrinking Δ, U_ν's and U_ν''s, one have that the Kodaira–Spencer map $\rho_{\mathrm{KS},\pi} \colon T_{T,t} \to H^1(X_t, T_{X_t})$ of the deformation family $\pi \colon \mathcal{X} \to T$ is injective for all $t \in T$.*

PROOF. Denote by α and β the maps $H^0(W_t^*, T_{W_t^*}) \to H^1(X_t, T_{X_t})$ and $H^1(X_t, T_{X_t}) \to H^1(M_t, T_{M_t}) \oplus H^1(M'_t, T_{M'_t})$, respectively, which appear in the Mayer–Vietoris sequence for the open covering $\{M_t, M'_t\}$ of X_t, where $W_t^* = M_t \cap M'_t$. It follows from Lemma 4.2 and Lemma 4.5 that there exist generators $\{\theta_1, \theta_2\}$ of $H^0(W_t^*, T_{W_t^*})$ such that $\rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial t_1}) = \alpha(\theta_1)$ and $\rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial t_2}) = \alpha(\theta_2)$ hold. Thus we have from Lemma 4.3 and the exactness of the sequence that these elements $\rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial t_1})$ and $\rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial t_2})$ are generators of the kernel of the map β. Denoting by $r_1\colon H^1(M_t, T_{M_t}) \oplus H^1(M'_t, T_{M'_t}) \to H^1(M_t, T_{M_t})$ the first projection and by $r_2\colon H^1(M_t, T_{M_t}) \oplus H^1(M'_t, T_{M'_t}) \to H^1(M'_t, T_{M'_t})$ the second projection, one can deduce from the argument in the proof of Proposition 4.7 that $\{r_1 \circ \beta \circ \rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial q_\nu})\}_{\nu=1}^8$ are linearly independent, $r_2 \circ \beta \circ \rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial q_\nu}) = 0$, $r_1 \circ \beta \circ \rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial q'_\nu}) = 0$, and that $\{r_2 \circ \beta \circ \rho_{\mathrm{KS},\pi}(\frac{\partial}{\partial q'_\nu})\}_{\nu=1}^8$ are linearly independent. Therefore we have that the images of $\frac{\partial}{\partial t_1}$, $\frac{\partial}{\partial t_2}$, $\frac{\partial}{\partial q_\nu}$'s and $\frac{\partial}{\partial q'_\nu}$'s by $\rho_{\mathrm{KS},\pi}$ are linearly independent, which shows the proposition. $\square$

In the above, we considered the deformation over an 18-dimensional space by fixing C_0, C'_0 and L_0. On the other hand, one can also consider the deformation obtained by changing C_0, C'_0 and L_0. However, at this moment we do not know the precise relationship between such a deformation and the above one over an 18-dimensional space.

QUESTION 4.13. What is the maximum dimension of a complex manifold T over which there exists a deformation family $\pi\colon \mathcal{X} \to T$ whose fibers are K3 surfaces obtained by the construction as in §3 such that the Kodaira–Spencer map $\rho_{\mathrm{KS},\pi}\colon T_{T,t} \to H^1(X_t, T_{X_t})$ is injective for all $t \in T$?

5 Proof of Theorem 1.1 and Corollary 1.2

5.1 Proof of Theorem 1.1

In this section, we prove Theorem 1.1. It is sufficient to show that the K3 surface X we constructed in §3 can be a non-Kummer surface if we choose nine points configurations appropriately. Main idea is to use the deformation family $P\colon \mathcal{X} \to T$ as in Proposition 4.7. As T is 8-dimensional manifold and the Kodaira–Spencer map of this family is injective, it is naturally expected that T covers at least one non-Kummer point in the K3 moduli. This heuristic observation is actually justified by the following proposition, which the author learned from Dr. Takeru Fukuoka.

PROPOSITION 5.1. *Let $P\colon \mathcal{X} \to T$ be a deformation family of K3 surfaces. Assume that the Kodaira–Spencer map $\rho_{\mathrm{KS,P}}\colon T_T \to R^1\pi_* T_{\mathcal{X}/T}$ is injective. Then there exists $t \in T$ with $\rho(X_t) \leq 20 - \dim T$, where $X_t := P^{-1}(t)$ and $\rho(X_t)$ is the Picard number of X_t.*

PROOF. Take a base point $0 \in T$ and denote by L the K3 lattice $H^2(X_0, \mathbb{Z})$. Fix a marking $R^2\pi_*\mathbb{C}_{\mathcal{X}} \cong (L_{\mathbb{C}})_T$, where $L_{\mathbb{C}} := L \otimes \mathbb{C}$. Consider a map $V_\bullet \colon T \to \mathbb{P}(L_{\mathbb{C}})$ defined by $t \mapsto V_t := H^0(X_t, K_{X_t})^\perp$ for each $t \in T$, where we are regarding $\mathbb{P}(L_{\mathbb{C}})$ as the set of hyperplanes of $L_{\mathbb{C}}$. It follows from Torelli's theorem that the map $V_\bullet$ is a locally closed embedding of T into $\mathbb{P}(L_{\mathbb{C}})$. Therefore $\mathrm{Image}\, V_\bullet$ is a locally closed subvariety of $\mathbb{P}(L_{\mathbb{C}})$ of dimension $\dim T$. Define $r \colon \mathbb{P}(L_{\mathbb{C}}) \to \mathbb{Z}$ by $r(V) := \mathrm{rank}(L \cap V)$. Note that $r(V_t) = \mathrm{rank}(H^2(X_t, \mathbb{Z}) \cap (H^{1,1}(X_t, \mathbb{C}) \oplus H^{0,2}(X_t, \mathbb{C}))) = \rho(X_t) + 1$ holds for each $t \in T$. Therefore the set $\{t \in T \mid \rho(X_t) < 21 - \dim T\}$ is rewritten as $V_\bullet^{-1}((\mathrm{Image}\, V_\bullet) \setminus \{V \in \mathbb{P}(L_{\mathbb{C}}) \mid r(V) \geq 22 - \dim T\})$. By Lemma 5.2 below, $\{V \in \mathbb{P}(L_{\mathbb{C}}) \mid r(V) \geq 22 - \dim T\}$ is a countable union of $(\dim T - 1)$-dimensional linear subspaces of $\mathbb{P}(L_{\mathbb{C}})$. Thus we obtain that $\{t \in T \mid \rho(X_t) < 21 - \dim T\} \neq \emptyset$. $\square$

LEMMA 5.2. *Let $r \colon \mathbb{P}(L_{\mathbb{C}}) \to \mathbb{Z}$ be as in the proof of Proposition 5.1. Then $F_n := \{V \in \mathbb{P}(L_{\mathbb{C}}) \mid r(V) \geq n\}$ is a countable union of $(21 - n)$-dimensional linear subspaces of $\mathbb{P}(L_{\mathbb{C}})$ for each $n = 0, 1, 2, \dots, 21$.*

PROOF. Set $\Lambda := \{M \subset L \mid M : \text{sub module},\ \mathrm{rank}\, M = n\}$. For $M \in \Lambda$ and $W \in \mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})$, it clearly holds that $p_M^{-1}(W) \in F_n$, where $M_{\mathbb{C}} := M \otimes \mathbb{C}$ and $p_M \colon L_{\mathbb{C}} \to L_{\mathbb{C}}/M_{\mathbb{C}}$ is the natural projection. Conversely, for each $V \in F_n$ and a sublattice $M \subset V$ of rank n, we have $V = p_M^{-1}(W)$ by defining $W := V/M_{\mathbb{C}} \in \mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})$. Therefore we obtain the description

$$F_n = \bigcup_{M \in \Lambda} \{p_M^{-1}(W) \mid W \in \mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}})\}.$$

As Λ is countable and the map $p_M^{-1}(-) \colon \mathbb{P}(L_{\mathbb{C}}/M_{\mathbb{C}}) \ni W \mapsto p_M^{-1}(W) \in F_n \subset \mathbb{P}(L_{\mathbb{C}})$ is a linear embedding for each M, the lemma follows. $\square$

Proof of Theorem 1.1. Let $P \colon \mathcal{X} \to T$ be the deformation family as in Proposition 4.12. Then it follows from Proposition 5.1 that one can take $t \in T$ such that $\rho(X_t) \leq 20 - 18 < 16$. As such X_t can never be a Kummer surface, the theorem follows. $\square$

5.2 Proof of Corollary 1.2

One can easily deduce the corollary by considering the universal covering of a leaf of a Levi-flat hypersurface $M_r \subset X$ in §3. $\square$

6 A relative variant of Arnol'd's Theorem

6.1 Proof of Theorem 4.6

In this section, we prove Theorem 4.6. Here we use the notations in §4.2. Fix a sufficiently fine open covering $\{U_j\}$ of C_0 with $\#\{U_j\} < \infty$ and a coordinate z_j of U_j such that $z_k = z_j + A_{kj}$ holds on each U_{jk} for some constant $A_{kj} \in \mathbb{C}$. As C_0 is an elliptic curve, we can take such coordinates by considering those induced by the natural coordinate of the universal cover $\mathbb{C}$. Fix also another open covering $\{U_j^*\}$ of C_0 with $\#\{U_j^*\} = \#\{U_j\}$ such that $U_j^* \Subset U_j$ for each j. In the following proof of Theorem 4.6, we use the following:

Lemma 6.1 (=[U, Lemma 4]). *Let M be a compact complex manifold, $\mathcal{U} = \{U_j\}_{j=1}^N$ an open covering of M, and $\mathcal{U}^* = \{U_j^*\}_{j=1}^N$ be an open covering of M such that $U_j^* \Subset U_j$ for each j. Then there exists a positive constant $K = K(M,\mathcal{U},\mathcal{U}^*)$ such that, for any flat line bundle $E \in \check{H}^1(\mathcal{U}, U(1))$ over M and for any 0-cochain $\mathfrak{f} =\in \check{C}^0(\mathcal{U}, \mathcal{O}_M(E))$, the inequality*

$$d(\mathbb{I}_M, E) \cdot \|\mathfrak{f}\| \leq K \cdot \|\delta\mathfrak{f}\|$$

holds. □

Here we denote by $\|\mathfrak{f}\|$ the value $\max_j \sup_{p\in U_j} |f_j(p)$ for each element $\mathfrak{f} = \{(U_j, f_j)\}_j$ of $\check{C}^0(\mathcal{U}, \mathcal{O}_M(E))$, and by $\|\mathfrak{g}\| := \max_{j,k} \sup_{p\in U_{jk}} |g_{jk}(p)|$ for each element $\mathfrak{g} = \{(U_{jk}, g_{jk})\}_{j,k}$ of $\check{C}^1(\mathcal{U}, \mathcal{O}_M(E))$.

In what follows, we always assume that T is a sufficiently small open ball centered at the base point $0 \in T$. As $N_{\mathcal{C}/\mathcal{S}} \cong \mathrm{Pr}_1^* L$, there exists $t_{jk} \in U(1)$ for each j and k such that $N_{\mathcal{C}/\mathcal{S}}^{-1} = [\{(U_{jk} \times T, t_{jk})\}] \in H^1(\mathcal{C}, \mathcal{O}_{\mathcal{C}}^*)$. Take a neighborhood $\mathcal{V}_j$ of $U_j \times T$ and a defining function w_j of $U_j \times T$ in $\mathcal{V}_j$. It is easily observed that we can choose w_j such that $t_{jk} w_k = w_j + O(w_j^2)$ holds on each $\mathcal{V}_{jk} := \mathcal{V}_j \cap \mathcal{V}_k$. By fixing a holomophic extension of the coordinate function z_j on U_j to $\mathcal{V}_j$, we first show the following lemma, which can be regarded as a relative variant of Ueda's theorem [U, Theorem 3] for elliptic curves.

Lemma 6.2. *By shrinking T and $\mathcal{V}_j$'s if necessary, one can take $\{(\mathcal{V}_j, w_j)\}$ such that $t_{jk} w_k = w_j$ holds on each $\mathcal{V}_{jk}$.*

Proof. By shrinking T if necessary, we can take a positive number $Q > 0$ such that $\{(z_j, w_j, t) \in \mathcal{V}_j \mid z_j \in U_j \cap U_k^*, |w_j| \leq Q^{-1}\} \subset \mathcal{V}_k$ for each j and k. Lemma 6.2 is shown by the same argument as in the proof of [U, Theorem 3]. We will construct a new defining function u_j of $C_0 \times T$ in $\mathcal{V}_j$ by solving a Schröder type functional equation

$$w_j = u_j + \sum_{\nu=2}^{\infty} f_{j|\nu}(z_j, t) \cdot u_j^\nu \tag{2}$$

on each $\mathcal{V}_j$, where the coefficient functions $\{f_{j|\nu}\}_{\nu=2}^{\infty}$ are constructed inductively just in the same manner as in [U, §4.2] so that the solution u_j satisfies $t_{jk}u_k = u_j$ on each $\mathcal{V}_{jk}$ if exists. Note that the Ueda's obstructions classes automatically vanish in our configurations, since $H^1(C_0 \times T, \mathrm{Pr}_1^* L^{-n}) = 0$ for each $n \geq 1$. Here we used the condition that L is non-torsion. Moreover, by $H^0(C_0 \times T, \mathrm{Pr}_1^* L^{-n}) = 0$, we have that each coefficient function $f_{j|\nu}$ is constructed uniquely as a holomorphic function on $U_j \times T$).

Therefore, all we have to do is to show the existence of the holomorphic solution u_j of the functional equation (2). By the implicit function theorem, it is sufficient to construct a convergent majorant series $A(u_j) = u_j + \sum_{\nu=2}^{\infty} A_\nu \cdot u_j^\nu$ for the functional equation (2). Such a majorant series $A(X)$ can be constructed by the same argument as in [U, §4.6] as the solution of the functional equation

$$\sum_{\nu=2}^{\infty} d(\mathbb{I}_{C_0}, L^{\nu-1}) \cdot A_\nu X^\nu = K \cdot \frac{M \cdot A(X)^2}{1 - M \cdot A(X)},$$

where $K = K(C_0, \{U_j\}, \{U_j^*\})$ is the constant as in Lemma 6.1 and M is a positive constant sufficiently larger than Q and $\max_j \sup_{\mathcal{V}_j} |w_j|$. By Siegel's technique [S] (see also [U, Lemma 5]), the solution $A(X)$ actually has a positive radius of convergence, which proves the lemma. □

In what follows, we always take a defining function w_j of $U_j \times T$ in $\mathcal{V}_j$ as in Lemma 6.2. Next we will show the existence of a suitable extension of the coordinate function $z_j \colon U_j \times T \to \mathbb{C}$ to $\mathcal{V}_j$. For clarity, we will denote (not by z_j as above, but) by $\zeta_j \colon \mathcal{V}_j \to \mathbb{C}$ the fixed extension of z_j in what follows. We will show the following:

Lemma 6.3. *By shrinking T and $\mathcal{V}_j$'s if necessary, one can take a holomorphic function $\zeta_j \colon \mathcal{V}_j \to \mathbb{C}$ such that $\zeta_j|_{U_j \times T} = z_j$ holds on on each $\mathcal{V}_j$ and $\zeta_k = \zeta_j + A_{kj}$ holds on each $\mathcal{V}_{jk}$.*

Proof. Fix a local projection $P_j \colon \mathcal{V}_j \to U_j \times T$ with $\pi|_{\mathcal{V}_j} = \mathrm{Pr}_2 \circ P_j$ for each j. We use a function $\zeta_j := P_j^* z_j$ as an initial extension function of $z_j \colon U_j \times T \to \mathbb{C}$ on each $\mathcal{V}_j$. In what follows, we denote by $g(\zeta_j, t)$ the function $P_j^* g$ for a function $g \colon U_j \to \mathbb{C}$. Then the expansion of ζ_k by w_j can be written as

$$\zeta_k = A_{kj} + \zeta_j + f_{kj}^{(1)}(\zeta_j, t) \cdot w_j + f_{kj}^{(2)}(\zeta_j, t) \cdot w_j^2 + \cdots$$

on each on $\mathcal{V}_{jk}$. As in the proof of the previous lemma, we will construct a new extension u_j of z_j by the defining equation

$$\zeta_j = u_j + \sum_{\nu=1}^{\infty} F_j^{(\nu)}(\zeta_j, t) \cdot w_j^\nu \tag{3}$$

on each $\mathcal{V}_j$.

First we explain how to define $\{F_j^{(1)}\}_j$. By adding three equations

$$\begin{aligned}\zeta_k &= A_{kj} + \zeta_j + f_{kj}^{(1)}(\zeta_j, t)\cdot w_j + O(w_j^2),\\ \zeta_\ell &= A_{k\ell} + \zeta_k + f_{\ell k}^{(1)}(\zeta_k, t)\cdot w_k + O(w_k^2),\end{aligned}$$

and

$$\zeta_j = A_{\ell j} + \zeta_\ell + f_{j\ell}^{(1)}(\zeta_\ell, t)\cdot w_\ell + O(w_\ell^2)$$

on $\mathcal{V}_{jk\ell}$ and by considering the coefficients of w_j in the both hands sides, we obtain the equality $f_{kj}^{(1)}(z_j,t) + f_{\ell k}^{(1)}(z_j,t)\cdot t_{jk}^{-1} + f_{j\ell}^{(1)}(z_j,t)\cdot t_{j\ell}^{-1} = 0$ on $U_{jk\ell}\times T$. Therefore we have that $[\{(U_{jk}\times T, f_{kj}^{(1)})\}]$ defines an element of $H^1(C_0\times T, \mathrm{Pr}_1^*L^{-1})$, which is equal to zero as a group. Thus we can take a holomorphic function $F_j^{(1)}$ on each $U_j\times T$ such that $F_j^{(1)} - t_{jk}^{-1}\cdot F_k^{(1)} = -f_{kj}^{(1)}$ holds on each $U_{jk}\times T$. As it follows from $H^0(C_0\times T, \mathrm{Pr}_1^*L^{-n}) = 0$ that such functions are unique, it gives the definition of $\{F_j^{(1)}\}$. Note that, by using these functions $\{F_j^{(1)}\}$, it clearly holds that the solution $\{u_j\}$ of the functional equation (3) satisfies $u_k - u_j = A_{kj} + O(w_j^2)$ on each $\mathcal{V}_{jk}$ after fixing $\{F_j^{(\nu)}\}_{\nu\geq 2}$ in any manner.

Next we explain how to define $F_j^{(\nu)}$ for $\nu > 1$ inductively. Assume that $\{F_j^{(\nu)}\}$ are already determined for each $\nu\leq n$ so that the following inductive assumption is satisfied.

(Inductive Assumption)$_n$: The solution $\{u_j\}$ of the functional equation (3) satisfies $u_k - u_j = A_{kj} + O(w_j^{n+1})$ on each $\mathcal{V}_{jk}$ after fixing $\{F_j^{(\nu)}\}_{\nu>n}$ in any manner.

Here we will construct $\{F_j^{(n+1)}\}$ such that (Inductive Assumption)$_{n+1}$ is satisfied. Let v_j be the solution of

$$\zeta_j = v_j + \sum_{\nu=1}^{n} F_j^{(\nu)}(\zeta_j, t)\cdot w_j^\nu.$$

Then, as we have that

$$\begin{aligned}-A_{kj} + v_k + \sum_{\nu=1}^{n} F_k^{(\nu)}\cdot w_k^\nu &= -A_{kj} + \zeta_k = \zeta_j + \sum_{\nu=1}^{n+1} f_{kj}^{(\nu)}\cdot w_j^\nu + O(w_j^{n+2})\\ &= \left(v_j + \sum_{\nu=1}^{n} F_j^{(\nu)}\cdot w_j^\nu\right) + \sum_{\nu=1}^{n+1} f_{kj}^{(\nu)}\cdot w_j^\nu + O(w_j^{n+2}),\end{aligned}$$

we obtain the inequality

$$v_k + \sum_{\nu=1}^{n} F_k^{(\nu)}(\zeta_k,t)\cdot w_k^\nu = v_j + A_{kj} + \sum_{\nu=1}^{n}\left(F_j^{(\nu)}(\zeta_j,t) + f_{kj}^{(\nu)}(\zeta_k,t)\right)\cdot w_j^\nu + f_{kj}^{(n+1)}(\zeta_k,t)\cdot w_j^{n+1} + O(w_j^{n+2}).$$

The coefficient of w_j^ν in the expansion of the left hand side can be calculated to be equal to $F_k(z_k,t)\cdot t_{jk}^{-\nu} + h_{kj}^{(\nu)}(z_j,t)$, where we denote by $h_{kj}^{(\nu)}(z_j,t)$ the function $\sum_{\mu=1}^{\nu-1} H_{kj,(\nu-\mu)}^{(\mu)}(z_j,t)\cdot t_{jk}^{-\mu}$ defined by using the coefficient functions $H_{kj,\lambda}^{(\nu)}$'s of the expansion

$$F_k^{(\nu)}(\zeta_k,t) = F_k^{(\nu)}(\zeta_k(\zeta_j,w_j,t),t) = F_k^{(\nu)}(z_k(z_j,0,t),t) + \sum_{\lambda=1}^{\infty} H_{kj,\lambda}^{(\nu)}(\zeta_j,t)\cdot w_j^\lambda$$

of $F_k^{(\nu)}$ by w_j on $\mathcal{V}_{jk}$. Note that $h_{jk,\nu}(\zeta_j, t)$ is determined only from $\{F_j^{(\mu)}\}_{\mu<\nu}$ and does not depend on the choice of $\{F_j^{(\mu)}\}_{\mu\geq\nu}$. Therefore, we obtain from (Inductive Assumption)$_n$ the equation

$$v_k = v_j + A_{kj} + \left(-h_{kj}^{(n+1)}(\zeta_j, t) + f_{kj}^{(n+1)}(\zeta_j, t)\right) \cdot w_j^{n+1} + O(w_j^{n+2})$$

on each $\mathcal{V}_{jk}$. By using this equation, it follows from just the same argument as in the definition of $\{F_j^{(1)}\}_j$ that there uniquely exists a holomorphic function $F_j^{(n+1)}$ on each $U_j \times T$ such that $F_j^{(n+1)} - t_{jk}^{-n-1} \cdot F_k^{(n+1)} = h_{kj}^{(n+1)} - f_{kj}^{(n+1)}$ holds on each $U_{jk} \times T$, by which we define $\{F_j^{(n+1)}\}$ (The assertion (Inductive Assumption)$_{n+1}$ is easily checked by construction).

Finally we show the convergence of the right hand side of the equation (3). We construct a convergent majorant series $A(X) = \sum_{\nu=1}^{\infty} A_\nu \cdot X^\nu$ for the series $\sum_{\nu=1}^{\infty} F_j^{(\nu)}(\zeta_j, t) \cdot X^\nu$. Take positive number M such that $\max_j \sup_{\mathcal{V}_j} |\zeta_j| < M$. Assume that $\{A_\nu\}_{\nu\leq n}$ satisfies $\max_j \sup_{U_j\times T} |F_j^{(\nu)}| \leq A_\nu$. Then, from the Cauchy–Riemann equality, it holds on each $U_j \cap U_k^*$ that

$$|h_{kj}^{(n+1)} - f_{kj}^{(n+1)}| \leq |f_{kj}^{(n+1)}| + \sum_{\nu=1}^{n} |H_{kj,(n+1-\nu)}^{(\nu)}| \leq MQ^{n+1} + \sum_{\nu=1}^{n} A_\nu Q^{n+1-\nu},$$

of which the right hand side is equal to the coefficient of X^{n+1} in the expansion of

$$M\sum_{\nu=1}^{\infty} Q^\nu X^\nu + \left(\sum_{\nu=1}^{\infty} A_\nu X^\nu\right) \cdot \left(\sum_{\lambda=1}^{\infty} Q^\lambda X^\lambda\right) = \frac{Q \cdot (M + A(X)) \cdot X}{1 - QX}.$$

From this observation and Lemma 6.1, it turns out that the series $A(X)$ defined by the functional equation

$$\sum_{n=1}^{\infty} d(\mathbb{I}_{C_0}, L^n) \cdot A_n X^n = 2K \cdot \frac{Q \cdot (M + A(X)) \cdot X}{1 - QX}$$

is a majorant series of the series $\sum_{\nu=1}^{\infty} F_j^{(\nu)}(\zeta_j, t) \cdot X^\nu$, where $K = K(C_0, \{U_j\}, \{U_j^*\})$ is the constant as in Lemma 6.1. Thus it is sufficient to show the solution $A(X)$ has a positive radius of convergence.

Define a new power series $B(X) = X + B_2 X^2 + B_3 X^3 + \cdots$ by $B(X) := X + X \cdot A(X)$ and $\widehat{B}(X) = X + \widehat{B}_2 X^2 + \widehat{B}_3 X^3 + \cdots$ by

$$\sum_{n=2}^{\infty} d(\mathbb{I}_{C_0}, L^{n-1}) \cdot \widehat{B}_n X^n = 2KQ \cdot \frac{(M+1) \cdot \widehat{B}(X)^2}{1 - Q\widehat{B}(X)}.$$

By Siegel's technique [S] (see also [U, Lemma 5]), it follows that $\widehat{B}(X)$ actually has a positive radius of convergence. As

$$\sum_{n=2}^{\infty} d(\mathbb{I}_{C_0}, L^{n-1}) \cdot B_n X^n = 2KQ \cdot \frac{(M \cdot X + B(X) - X) \cdot X}{1 - QX},$$

we can show by the simple inductive argument that $\widehat{B}_\nu \geq B_\nu (= A_{\nu-1})$ for each $\nu \geq 2$, which proves the lemma. $\square$

Proof of Theorem 4.6. Take a coordinate system (z_j, w_j, t) of each $\mathcal{V}_j$ such that $\{w_j\}$ is as in Lemma 6.2 and $\{\zeta_j\}$ is as in Lemma 6.3. Define a map $P\colon \bigcup_j \mathcal{V}_j \to C_0 \times T$ by $p(\zeta_j, w_j, t) := (\zeta_j, t) \in U_j \times T$ on each $\mathcal{V}_j$, which is well-defined by Lemma 6.3. By regarding w_j's as fiber coordinates, we can naturally regard $\bigcup_j \mathcal{V}_j$ as a open neighborhood of the zero-section of $N_{\mathcal{C}/\mathcal{S}}$, which proves the theorem. $\square$

6.2 More generalized variant

Theorem 4.6 can be shown not only in the case where $\mathcal{C} \cong C_0 \times T$ and $\pi|_{\mathcal{C}} = \mathrm{Pr}_2$ hold, but also in the case where $\pi|_{\mathcal{C}}\colon \mathcal{C} \to T$ is a proper holomorphic submersion whose fibers $C_t := S_t \cap \mathcal{C}$ are elliptic curves:

THEOREM 6.4. *Let $\pi\colon \mathcal{S} \to T$ be a deformation family of complex surfaces over a ball in $\mathbb{C}^n$, and $\mathcal{C} \subset \mathcal{S}$ be a submanifold such that $\pi|_{\mathcal{C}}$ is a deformation family of smooth elliptic curves. Assume that $d(\mathbb{I}_{C_t}, N^n_{C_t/S_t})$ does not depend on $t \in T$ for each n and that the Diophantine condition $-\log d(\mathbb{I}_{C_t}, N^n_{C_t/S_t}) = O(\log n)$ holds as $n \to \infty$. Then, by shrinking T if necessary, there exists a tubular neighborhood $\mathcal{W}$ of $\mathcal{C}$ in $\mathcal{S}$ which is isomorphic to a neighborhood of the zero section in $N_{\mathcal{C}/\mathcal{S}}$.*

Note that, we have to choose the invariant distance d of each $\mathrm{Pic}^0(C_t)$ appropriately in order it to satisfy the condition that $d(\mathbb{I}_{C_t}, N^n_{C_t/S_t})$ does not depend on $t \in T$ for each n. A typical example of the configuration is as follows.

EXAMPLE 6.5. Let $\tau(t)$ be a point in the upper half plane such that $C_t \cong \mathbb{C}/\langle 1, \tau(t)\rangle$. By choosing $\tau(t)$'s appropriately, we may assume that τ is a holomorphic function. By regarding $\mathrm{Pic}^0(C_t)$ as C_t via the isomorphism $C_0 \ni p \mapsto \mathcal{O}_{C_t}(p - [0]) \in \mathrm{Pic}^0(C_t)$, we define an invariant distance d of each $\mathrm{Pic}^0(C_t)$ by

$$d([0], [\alpha + \beta \cdot \tau(t)]) := \min\{|\alpha|, |1-\alpha|\} + \min\{|\beta|, |1-\beta|\}$$

for each $0 \leq \alpha, \beta < 1$, where $[z]$ is the image of $z \in \mathbb{C}$ by the covering map $\mathbb{C} \to \mathbb{C}/\langle 1, \tau(t)\rangle \cong C_t$. Take two algebraic irrational numbers α and β. Define a divisor $\mathcal{D}$ of $\mathcal{C}$ by $\mathcal{D} \cap C_t = [\alpha + \beta \cdot \tau(t)] - [0]$. Then, if the normal bundle $N_{\mathcal{C}/\mathcal{S}}$ is the line bundle corresponding to $\mathcal{D}$, then $d(\mathbb{I}_{C_t}, N^n_{C_t/S_t})$ does not depend on $t \in T$ for each n and that the Diophantine condition $-\log d(\mathbb{I}_{C_t}, N^n_{C_t/S_t}) = O(\log n)$ holds as $n \to \infty$.

We can prove Theorem 6.4 by almost the same manner as the proof of Theorem 4.6. Only the difficulty is the t-dependence of the constant K as in Lemma 6.1. In order to overcome this difficulty, we use the following:

LEMMA 6.6. *Assume that each U_j is a coordinate open ball. Then one can take a constant $K = K(M,\mathcal{U},\mathcal{U}^*)$ as in Lemma 6.1 such that K depends only on the number $N = \#\mathcal{U}$ and the maximum of the radii of U_j^*'s calculated by using the Kobayashi metrics of U_j's.*

Lemma 6.6 follows directly from the improved proof of Lemma 6.1 we will describe in §6.3, which the author learned from Prof. Tetsuo Ueda.

Proof of Theorem 6.4. Fix a sufficiently fine open covering $\{U_j\}$ of C_0 with $\#\{U_j\} < \infty$ and a coordinate z_j of U_j such that $z_k = z_j + A_{kj}$ holds on each U_{jk} for some constant $A_{kj} \in \mathbb{C}$. We may assume that each U_j is a coordinate open ball. Fix also another open covering $\{U_j^*\}$ of C_0 with $\#\{U_j^*\} = \#\{U_j\}$ such that $U_j^* \Subset U_j$ for each j. Then, by shrinking T if necessary, we can regard $\mathcal{C}$ as a complex manifold which is obtained by patching $U_j \times T$'s (or $U_j^* \times T$'s) by using the coordinate transformations in the form of $z_k = z_j + A_{kj}(t)$, where A_{kj} is a holomorphic function defined on T with $A_{kj}(0) = A_{kj}$. It follows from Lemma 6.6 that the constant K as in Lemma 6.1 can be taken as a constant which is independent of the parameter $t \in T$. Then we can carry out the same argument as in the previous subsection to obtain Theorem 6.4. □

6.3 An alternative proof of Ueda's lemma with effective constant K

Here we describe a simple proof of Lemma 6.1, which the author learned from Prof. Tetsuo Ueda. One of the most remarkable points in this proof is that the constant $K = K(M,\mathcal{U},\mathcal{U}^*)$ can be described explicitly. Actually, we will construct the constant K so that the inequality

$$K < 1 + 2 \cdot \left(\frac{2}{1-s}\right)^{N+2}$$

holds, where s is the maximum of the constants s_j's in the following:

LEMMA 6.7. *Assume that each U_j is a coordinate open ball. For each j, there exists a positive constant s_j less than 1 which satisfies the following assertion: For any holomorphic function $f\colon U_j \to \mathbb{C}$ with $\sup_{z\in U_j} |f(z)| < 1$, if there exists a point $z_0 \in U_j^*$ with $f(z_0) = 0$, then it holds that $\sup_{z\in U_j^*} |f(z)| < s_j$. Moreover, we can take such s_j so that it depends only on the radius of U_j^* calculated by using the Kobayashi metric of U_j.*

PROOF. Lemma follows from the Schwarz–Pick theorem-type property of the Kobayashi metric. □

Set $L_1 := \frac{2s}{1-s}$ and $L_2 := \frac{1+s}{1-s}$. Then we have the following:

LEMMA 6.8. *For any holomorphic function $f\colon U_j \to \mathbb{C}$ with $\sup_{z\in U}|f(z)|<1$ and for any points $z_1, z_2 \in U_j^*$, we have the inequalities $|f(z_1)-f(z_2)| \leq L_1\cdot(1-|f(z_1)|)$ and $1-|f(z_2)| \leq L_2 \cdot (1-|f(z_1)|)$.*

PROOF. Set $a := f(z_1)$ and consider the Möbius transformation $T(w) := \frac{w-a}{1-\overline{a}w}$. As $T\circ f\colon U_j \to \Delta$ maps the point $z_1 \in U_j^*$ to 0, it follows from Lemma 6.7 that the modulus $|\zeta|$ of $\zeta := T\circ f(z_2)$ is less than s ($\Delta \subset \mathbb{C}$ is the unit disc). Therefore we have

$$|f(z_1)-f(z_2)| = \left|a - \frac{\zeta+a}{1+\overline{a}\zeta}\right| = \frac{(1-|a|^2)|\zeta|}{|1+\overline{a}\zeta|} < \frac{(1+|a|)s}{|1+\overline{a}\zeta|}\cdot(1-|a|) < \frac{2s}{1-s}\cdot(1-|a|),$$

which proves the first inequality.

The second inequality holds obviously when $a=0$ holds. When $a\neq 0$, let us consider the constant $\alpha := \frac{a}{|a|}$. Then, as it holds that $1 = |\alpha - T^{-1}(\zeta) + T^{-1}(\zeta)| \leq |\alpha - T^{-1}(\zeta)| + |T^{-1}(\zeta)|$, we have

$$1-|f(z_2)| \leq |\alpha - T^{-1}(\zeta)| = \left|\frac{a}{|a|} - \frac{\zeta+a}{1+\overline{a}\zeta}\right| \leq \frac{|a|+|a|\cdot|\zeta|}{|1+\overline{a}\zeta|}\cdot(1-|a|) \leq \frac{1+s}{1-s}\cdot(1-|a|),$$

which proves the second inequality. □

Denote by K_1 the constant $L_1\cdot L_2\cdot(L_2+1)^N$, and by K_2 the constant $L_2\cdot(L_2+1)^N$. Then we have the following:

LEMMA 6.9. *For each j, points $p, p' \in U_j^*$, and any 0-cochain $\mathfrak{f} = \{(U_j, f_j)\}_j \in \check{C}^0(\mathcal{U}, \mathcal{O}_M(E))$ with $\|\mathfrak{f}\|$=1, the inequalities $|f_j(p)-f_j(p')| \leq K_1\cdot\|\delta\mathfrak{f}\|$ and $1-|f_j(p)| \leq K_2\cdot\|\delta\mathfrak{f}\|$ hold.*

PROOF. Take a positive constant ε (slightly) larger than $\|\delta\mathfrak{f}\|$. Then $|f_{j_0}(p_0)| > 1-(\varepsilon - \|\delta\mathfrak{f}\|)$ holds for some $p_0 \in U_{j_0}$. Take a chain of open sets $U_{j_1}^*, U_{j_2}^*, \dots, U_{j_m}^*$ such that $p_0 \in U_{j_1}^*$ and that $U_{j_\mu}^* \cap U_{j_{\mu+1}}^* \neq \emptyset$ holds for each $1\leq \mu < m$. We shall show the following assertion: for each $p, p' \in U_{j_m}^*$, the inequality $|f_{j_m}(p) - f_{j_m}(p')| \leq L_1\cdot L_2\cdot(L_2+1)^m\cdot\varepsilon$ and $1-|f_{j_m}(p)| \leq L_2\cdot(L_2+1)^m\cdot\varepsilon$ hold. Note that, as any U_j can be linked with U_{j_0} by such a chain with length at most $N = \#\mathcal{U}$, Lemma 6.9 follows from this assertion.

The proof is by induction on m. First, we show the case of $m=1$. As $|f_{j_0}(p_0)| = |t_{j_1j_0}f_{j_0(p_0)}| \leq |t_{j_1j_0}f_{j_0}(p_0) - f_{j_1}(p_0)| + |f_{j_1}(p_0)| \leq \|\delta\mathfrak{f}\| + |f_{j_1}(p_0)|$ holds, it follows from Lemma 6.8 that

$$1-|f_{j_1}(p)| \leq L_2\cdot(1-|f_{j_1}(p_0)|) \leq L_2\cdot(1-|f_{j_0}(p_0)|+\|\delta\mathfrak{f}\|) < L_2\cdot((\varepsilon-\|\delta\mathfrak{f}\|)+\|\delta\mathfrak{f}\|) = L_2\cdot\varepsilon$$

holds for any $p\in U_{j_1}^*$. Thus the second inequality follows. By Lemma 6.8,

$$|f_{j_1}(p)-f_{j_1}(p')| \;\leq\; L_1\cdot(1-|f_{j_1}(p)|) \leq L_1\cdot L_2\cdot\varepsilon < L_1\cdot L_2\cdot(L_2+1)\cdot\varepsilon$$

holds for each $p, p' \in U^*_{j_1}$, from which we have the first inequality.

Next we show the case of $m \geq 2$ by assuming the assertion for $\mu < m$. Fix a point $p_m \in U^*_{m-1} \cap U^*_m$ and take any $p \in U^*_m$. Then, by the inductive assumption and the inequality $|f_{j_{m-1}}(p_m)| \leq |t_{j_m j_{m-1}} f_{j_{m-1}}(p_m) - f_{j_m}(p_m)| + |f_{j_m}(p_m)| \leq \|\delta\mathfrak{f}\| + |f_{j_m}(p_m)|$, we have that

$$\begin{aligned} 1 - |f_{j_m}(p)| &\leq L_2 \cdot (1 - |f_{j_m}(p_m)|) \leq L_2 \cdot (1 - |f_{j_{m-1}}(p_m)| + \|\delta\mathfrak{f}\|) \\ &\leq L_2 \cdot (L_2 \cdot (L_2+1)^{m-1} \cdot \varepsilon + \|\delta\mathfrak{f}\|) < L_2 \cdot (L_2+1)^m \cdot \varepsilon, \end{aligned}$$

from which the second inequality follows. The first inequality follows from this inequality and Lemma 6.8. □

Set $K := \max\{1+2K_1+2K_2, 2K_2\} (= 1+2K_1+2K_2)$. We shall prove that this constant K satisfies the property as in Lemma 6.1. Here we will use the invariant distance d as in [U, §4.5]: i.e.

$$d(\mathbb{I}_M, E) := \min_{\{(U_j, t_j)\}_j \in \check{C}^0(\mathcal{U}, U(1))} \max_{j,k} |t_{jk} \cdot t_k - t_j|,$$

where $\{t_{jk}\} \subset U(1)$ is such that $E = \{(U_{jk}, t_{jk})\} \in \check{Z}^1(\mathcal{U}, U(1))$. Note that $d(\mathbb{I}_M, E) \leq 2$ follows by definition for any E.

We may assume that $\|\mathfrak{f}\| = 1$. When $\|\delta\mathfrak{f}\| \geq K_2^{-1}$, we have that

$$d(\mathbb{I}_M, E) \cdot \|\mathfrak{f}\| \leq 2 \leq 2K_2 \cdot \|\delta\mathfrak{f}\|.$$

Therefore it is sufficient to show the Lemma by assuming that $\|\delta\mathfrak{f}\| < K_2^{-1}$. Take $t_{jk} \in U(1)$ such that $E = \{(U_{jk}, t_{jk})\} \in \check{Z}^1(\mathcal{U}, U(1))$ and fix points $q_j \in U^*_j$ and $q_{jk} \in U^*_j \cap U^*_k$. By the assumption and Lemma 6.9, we have that $1 - |f_j(q_j)| \leq K_2 \cdot \|\delta\mathfrak{f}\| < 1$. Therefore $f_j(q_j) \neq 0$ for each j. Set $t^*_j := \frac{f_j(q_j)}{|f_j(q_j)|}$. Then we have that

$$\begin{aligned} |t_{jk} t^*_k - t^*_j| \leq & \left| t_{jk} \left(\frac{f_k(q_k)}{|f_k(q_k)|} - f_k(q_k) \right) \right| + |t_{jk}(f_k(q_k) - f_k(q_{jk}))| + |t_{jk} f_k(q_{jk}) - f_j(q_{jk})| \\ & + |f_j(q_{jk}) - f_j(q_j)| + \left| f_j(q_j) - \frac{f_j(q_j)}{|f_j(q_j)|} \right| \\ \leq & (1 - |f_k(q_k)|) + |f_k(q_k) - f_k(q_{jk})| + \|\delta\mathfrak{f}\| + |f_j(q_{jk}) - f_j(q_j)| + (1 - |f_j(q_j)|) \end{aligned}$$

holds. Thus Lemma follows from the definition of our invariant distance and Lemma 6.9. □

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